

Turning the Liar Paradox into a Metatheorem of Basic Logic

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Abstract

We show that self-reference can be formalized in Basic logic by means of the new connective: @, called “entanglement”. In fact, the property of non-idempotence of the connective @ is a metatheorem, which states that a self-entangled sentence loses its own identity. This prevents having self-referential paradoxes in the corresponding metalanguage. In this context, we introduce a generalized definition of self-reference, which is needed to deal with the multiplicative connectives of substructural logics.

Keywords: substructural logics, metalanguage, object-language, self-reference, quantum computing

1 Introduction

Since Epimenides, the Greek philosopher who lived about 600 BC, the Liar paradox: “All Creteans are liars” (in its modern version: “This sentence is false”) remained “unsolved”.

In our opinion, the “paradox” arises because logicians have always tried to formalize self-reference within structural logics, which are not adequate to describe such a strong logical correlation, which resembles quantum entanglement. However, the appropriate connective was missing, until we found it in a recent paper [1] and called it @ = “entanglement”. The introduction of the connective @ is possible only in a substructural logic obtained by reflection of a metalanguage that is the mathematical formalism of quantum computing. In [1] we showed that such a logic is Basic logic [2].

Basic logic can accommodate the connective for quantum superposition ($\&$ = “and”) and the new connective for quantum entanglement (@ = “entanglement”).

The connective $\&$ in Basic logic can describe in fact quantum superposition because Basic logic is a paraconsistent logic [3].

The definition of @ is possible because Basic logic is a substructural logic (it has neither the contraction rule, nor the weakening rule), and is non-distributive.

As quantum superposition together with quantum entanglement lead to massive quantum parallelism that is the source of quantum computational speed-up [4], we argued [1] that Basic logic should be the most adequate logic for quantum computing. Also, we noticed that the absences of the contraction rule and of the weakening rule correspond, in quantum computing, to the no-cloning [5] and no-erase [6] theorems respectively.

Among the properties of the connective @, we found that @ is non-idempotent. The non-idempotence of @ is strictly related to the physical fact that self-entanglement is meaningless, unless one could clone the original qubit, which is impossible because of the no-cloning theorem.

And as we already mentioned, the quantum no-cloning theorem reflects itself into logic as the absence of the contraction rule (data cannot be copied).

Although at a first sight the non-idempotence of @ might seem a quite harmless property, in fact it is a metatheorem with intriguing consequences.

In this paper we prove the theorem, and explore its consequences. The theorem states that a self-entangled sentence cannot simply exist, as if it did, it would lose its identity. This becomes apparent when the metalanguage is reflected into a substructural logic like Basic logic, once the latter is equipped with the connective @.

On the other hand, it is a fact that self-referential sentences and paradoxes do appear in natural language, but this happens when the latter is the only metalanguage (with no hierarchy) and is reflected into a structural logic, or more generally, into a logic which cannot include the connective @.

Section 2 is a short review of some results of Ref. [1], namely, the definitions of quantum superposition and quantum entanglement in logical terms, and, in particular, the definition of the connective @.

Section 3 consists of the statement and interpretation of the no-self-reference metatheorem of Basic logic, which is based on the property of non-idempotence of the connective @. Then, we relate our results to the “Liar paradox”.

In Section 4, we introduce a new, generalized definition of self-reference, which can deal with substructural logics, and which reduces to the standard one in the case of structural logics.

In the Appendix A, we give the formal proof of the metatheorem discussed in Section 3.

2 The Logical Connective for Quantum Entanglement

The qubit is the unit of quantum information [7]:

$$|Q\rangle = a|0\rangle + b|1\rangle \quad (1)$$

where $\{|0\rangle, |1\rangle\}$ is called the computational basis and a, b are complex numbers called probability amplitudes such that $|a|^2 + |b|^2 = 1$. Two qubits are said entangled when the bipartite state $|Q\rangle_{AB}$ is not-separable, i.e.

$$|Q\rangle_{AB} \neq |Q\rangle_A \otimes |Q\rangle_B ,$$

where \otimes is the tensor product in Hilbert spaces. When the composite system of two qubits is in a non-separable state, it is impossible to attribute to each qubit a pure state, as their states are superposed with one another.

In particular, a bipartite state of two qubits is maximally entangled when it is one of the four Bell’s states [8]:

$$\begin{aligned} |\Phi_{\pm}\rangle_{AB} &= \frac{1}{\sqrt{2}} (|0\rangle_A \otimes |0\rangle_B \pm |1\rangle_A \otimes |1\rangle_B) , \\ |\Psi_{\pm}\rangle_{AB} &= \frac{1}{\sqrt{2}} (|0\rangle_A \otimes |1\rangle_B \pm |1\rangle_A \otimes |0\rangle_B) . \end{aligned} \quad (2)$$

For simplicity, in this paper we will consider only Bell states.

As we showed in [1], expressing the qubit $|Q\rangle_A$ in (1) in logical terms leads to the compound proposition:

$$Q_A \doteq A \& A^{\perp} \quad (3)$$

where the atomic proposition A is associated with bit $|1\rangle$, its primitive negation A^\perp is associated with bit $|0\rangle$, and the (right) connective $\&$ = “and” is the additive conjunction [2].

In the same way, the second qubit $|Q_B\rangle$ is expressed, in logical terms, by a second compound proposition $Q_B \doteq B \& B^\perp$.

Bell’s states will be expressed, in logical terms, by the expression

$$Q_{Bell} = Q_A @ Q_B \quad (4)$$

where @ is the new logical connective called “entanglement” [1].

The logical structure for, say, the Bell’s states $|\Phi_\pm\rangle_{AB}$ (which was given in [1] as the definition of @) is:

$$Q_A @ Q_B \doteq (A \wp B) \& (A^\perp \wp B^\perp) \quad (5)$$

where the (right) connective \wp = “par”, introduced by Girard in Linear logic [9] is the multiplicative disjunction, the dual of the (left) connective \otimes = “times”, which is the multiplicative conjunction. The logical rules for @, as well as its properties, can be found in [1]. In what follows, we will discuss the non-idempotence of @, also in relation with self-reference, and with the Liar paradox.

3 Non-idempotence of @ and the Liar Paradox

Here we give an informal proof of the non-idempotence of @ (a more formal proof will be given in the Appendix A). We want to prove:

$$Q_A @ Q_A \neq Q_A \quad (6)$$

From the definition of @ in (5), by replacing B with A we get:

$$Q_A @ Q_A \doteq (A \wp A) \& (A^\perp \wp A^\perp) \neq Q_A \quad (7)$$

as \wp is non-idempotent. In fact, to prove the idempotence of \wp would require the validity of both the contraction and weakening rules.

If one makes the formal proof one has to go both ways: to show that $A \vdash A \wp A$ does not hold because of the absence of the weakening rule, and that $A \wp A \vdash A$ does not hold because of the absence of the contraction rule.

If instead weakening and contraction did hold, then $\wp \equiv \vee$ ($\otimes \equiv \&$), and from the definition of @ we would get:

$$Q_A @ Q_A \doteq (A \vee A) \& (A^\perp \vee A^\perp) = Q_A \quad (8)$$

because of the idempotence of \vee . In that case, from the definition of the dual § of @ given in [1], namely:

$$Q_A \S Q_B \doteq (A \otimes B) \vee (A^\perp \otimes B^\perp) \quad (9)$$

we will also get:

$$Q_A \S Q_A \doteq (A \& A) \vee (A^\perp \& A^\perp) = Q_A \quad (10)$$

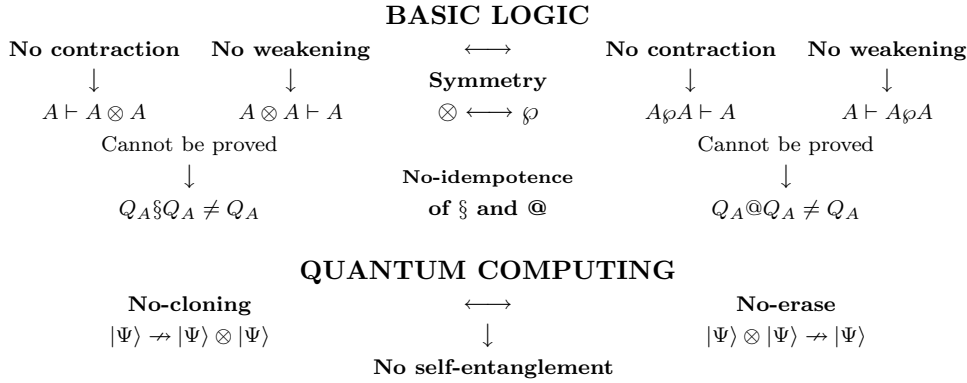
because of the idempotence of $\&$.

Notice, in particular, that the formal proof that the dual of \wp , namely \otimes = “times” is non-idempotent, would exchange the roles of the contraction and the weakening rules used in the proof

done for \wp . The fact that \otimes is non-idempotent, leads to the result that the dual of $@$, namely \S , is non-idempotent either. Then, $@$ (\S) is non-idempotent because $\wp(\otimes)$ is non-idempotent.

This illustrates an obvious physical fact: self-entanglement (entanglement of a qubit with itself) is impossible as it would require a quantum clone, which is forbidden by the no-cloning theorem. In a sense, one can say that the two main no-go theorems of quantum computing, namely the no-cloning and no-erase theorems are (logically) dual to each other. And the no-self-entanglement “corollary” is a consequence of the first one, when entanglement is expressed in terms of \S , and a consequence of the second one, when entanglement is expressed by $@$.

On the other hand, it turns out that the meaning of “no-self-entanglement” is much more profound in logic. In fact, affirming that in a certain formal language it is impossible to get a (compound) proposition (maximally) entangled with itself means that the language under study does not lead to self-referential sentences in the corresponding metalanguage. Schematically:



Notice that the property of the non-idempotence of $@$ (and of its dual \S) deals with the object language. However, when this property is translated into a metalanguage like natural language, we get:

$$\underbrace{Q_A}_{\text{a sentence entangled with}} \underbrace{@}_{\text{itself}} \underbrace{Q_A}_{\text{itself}} \neq \underbrace{Q_A}_{\text{itself}}$$

Then we state the following.

Metatheorem 1 *A sentence Q_A logically entangled with itself is not itself.*

This just expresses the impossibility of having self-entangled sentences in the metalanguage. It is not a paradox. However, many classical “paradoxes” like the Liar paradox: “This sentence is false” look very much like the property of the non-idempotence of $@$, which instead is a metatheorem. The reason is that in our classical reasoning, the concept of entanglement is missing. Moreover, when we try to formalize the “paradox” in any other logic (but Basic logic) which is lacking in the connective $@$, we fail. See Fig.1.

4 A generalized definition of self-reference

Self-reference, as usually understood, can be interpreted as a function F from the object-language L_0 to the metalanguage L_m , that is:

$$F: L_0 \rightarrow L_m \quad (11)$$

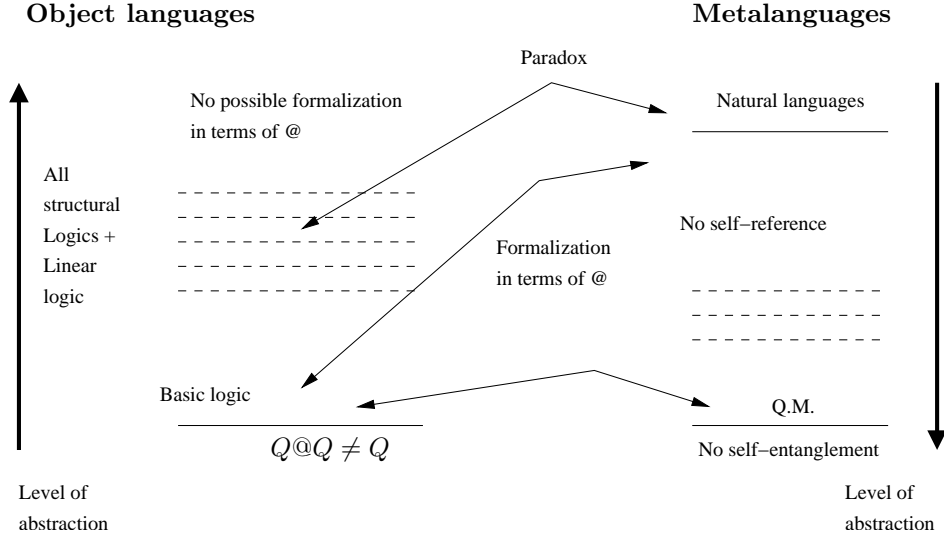


Figure 1: The Liar “Paradox” Revisited

such that

$$S \equiv F("S") \quad (12)$$

where $S \in L_m$ is a sentence in the metalanguage, and $"S" \in L_0$, is the name of S in the object-language. A self-referential sentence of kind (12) is, for example, the Liar sentence L : “This sentence is false”, i.e.

$$L \equiv \neg \text{True}("L") \quad (\text{with: } F \equiv \neg \text{True}) \quad (13)$$

By the use of Tarski’s definition of truth [10]:

$$\text{True}("A") \equiv A \quad (14)$$

Which is an axiom schema for an arbitrary formula A , the Liar sentence (13) becomes:

$$L \equiv \neg \text{True}("L") \equiv \neg L$$

leading to the self-referential paradox:

$$L \equiv \neg L \quad (15)$$

Here, we introduce a more general definition of self-reference:

$$S \equiv F(f("S")) \quad (16)$$

Where f is a function from the object-language into itself:

$$f: L_0 \rightarrow L_0$$

The function f is built as follows. Let us consider the name of S , i.e., $"S"$ in the object-language ($"S" \in L_0$), let us duplicate it ($"S", "S"$) by the “diagonal” function Δ :

$$\Delta: L_0 \rightarrow L_0 \times L_0$$

If the name of S is a formula of L_0 , (let us call S_0) we can send the pair (S_0, S_0) into a new formula $S_0 \bullet S_0$ by the binary connective \bullet

$$(S_0, S_0) \xrightarrow{\bullet} S_0 \bullet S_0$$

Then, f is the composite function $f = \bullet \circ \Delta$

$$f: L_0 \xrightarrow{\Delta} L_0 \times L_0 \xrightarrow{\bullet} L_0$$

There are two cases:

1. The connective \bullet is idempotent (for example: $\bullet \equiv \& = \text{"and"} , \bullet \equiv \vee = \text{"or"})$:

$$S_0 \bullet S_0 = S_0$$

That is, the function f has a fixed point:

$$f(\text{"S"}) = \text{"S"} \quad (17)$$

In this case, (16), by the use of (17), reduces to (12), namely, to the standard definition of self-reference.

2. The connective \bullet is non-idempotent (for example: $\bullet \equiv \otimes = \text{"times"} , \bullet \equiv \wp = \text{"par"} , \bullet \equiv @ = \text{"entanglement"})$:

$$S_0 \bullet S_0 \neq S_0$$

That is, the function f has no fixed point:

$$f(\text{"S"}) = \widehat{\text{"S"}} \neq \text{"S"} \quad (18)$$

In this case, (16) cannot be rewritten as (12). Notice that this fact is peculiar of sub-structural logics, like Basic logic and linear logic, which can accommodate multiplicative connectives like \otimes , \wp and $@$ (more precisely, Basic logic can accommodate all the three of them, and linear logic only two of them, namely \otimes and \wp). In structural logics like quantum logic, intuitionistic logic, and classical logic, \otimes “collapses” to $\&$, which is idempotent, \wp “collapses” to \vee , which is also idempotent, and $@$ “collapses” to a function of $(\vee, \&)$, which is idempotent as well.

In summary, in sub-structural logics, one is allowed to adopt the generalized version (16) of self-reference, while in structural logics, (16) reduces to (12), and one has to stick with the standard definition of self-reference.

In case 2., (16), by the use of (18), becomes:

$$S \equiv F(\widehat{\text{"S"}}) \quad (19)$$

In particular, if S is the Liar sentence L , and $F \equiv \neg \text{True}$, we get, from (19):

$$L \equiv \neg \text{True}(\widehat{L}) \quad (\text{with: } \widehat{L} \neq L) \quad (20)$$

Equation (20), by the use of Tarski’s definition of truth (14) becomes:

$$L \equiv \neg \text{True}(\widehat{L}) \equiv \neg \widehat{L} \quad (21)$$

hence:

$$L = \neg \hat{L} \quad (22)$$

which is not a paradox.

Let us consider now the composite function $F(f("S"))$ which appears in the generalized definition of self-reference (16), and let us call it σ .

$$\sigma: L_0 \rightarrow L_m \quad (23)$$

See the following diagram:

$$\begin{array}{ccc} L_0 & & \\ \downarrow F & \swarrow f & \\ L_m & \xleftarrow{\sigma} & L_0 \end{array}$$

$$f \circ F = \sigma \quad (24)$$

If the connective \bullet is idempotent, the interpretation of $(f \circ F)("S")$ is S in the metalanguage, and we get:

$$F \equiv \sigma \quad (25)$$

There are only two possible cases of study: linear logic with \otimes (it is sufficient to consider only \otimes , as \wp is the dual of \otimes), and Basic logic, with both \otimes and $@$.

The connective \otimes in the object-language is the “reflection” of the (physical) tensor product \otimes (where the same symbol is used) in the quantum mechanical metalanguage.

The fact that a multiplicative connective \bullet in the object-language is non-idempotent, is a necessary but not sufficient condition for having a generalized version of self-reference. The sufficient condition requires that the corresponding (physical) link in the metalanguage does never allow cloning. This is not the case of the tensor product \otimes , because one can clone a basis state:

$$|0\rangle \rightarrow |0\rangle \otimes |0\rangle, \quad |1\rangle \rightarrow |1\rangle \otimes |1\rangle \quad (26)$$

by means of the reversible XOR, or controlled NOT (CNOT) quantum logic gate. The CNOT gate operates on two qubits a and b . See the following diagram:

$$\begin{array}{ccc} a & \xrightarrow{\text{“control”}} \bullet & a \\ & \downarrow & \\ b & \xrightarrow{\text{“target”}} \oplus & a \oplus b \end{array} \quad (27)$$

The \oplus symbol in the diagram (27) represents modulo 2 addition. The CNOT flips the “target” input if its “control” input is $|1\rangle$, and does nothing if it is $|0\rangle$. So, one can clone a basis state, in the two cases:

$$\begin{cases} \text{control } |0\rangle \\ \text{target } |0\rangle \end{cases} \xrightarrow{XOR} |00\rangle \quad \begin{cases} \text{control } |1\rangle \\ \text{target } |0\rangle \end{cases} \xrightarrow{XOR} |11\rangle$$

(where $|00\rangle$ and $|11\rangle$ stand for the tensor products $|0\rangle \otimes |0\rangle$ and $|1\rangle \otimes |1\rangle$ respectively).

We recall that the basis states are, in the corresponding object-language, the atomic formulas $A = |1\rangle$, $A^\perp = |0\rangle$. Then, (26) does not hold in the object-language, as the connective $\otimes = \text{“times”}$ is non-idempotent:

$$A \otimes A \neq A, \quad A^\perp \otimes A^\perp \neq A^\perp.$$

But when one tries to clone a superposed state, for example the “cat” state:

$$|Q\rangle_{Cat} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

by means of the CNOT, one gets a Bell state, which is maximally entangled:

$$\begin{cases} \text{control } \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ \text{target } |0\rangle \end{cases} \xrightarrow{XOR} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

In fact, by the no-cloning theorem, it is forbidden to copy an unknown quantum state:

$$|Q\rangle \nrightarrow |Q\rangle \otimes |Q\rangle \quad (28)$$

Equation (28) holds in the object language as well:

$$Q \otimes Q \neq Q$$

Nevertheless, the problem with the connective “times” does appear in the case of basis states, as the corresponding physical link, the tensor product, can form a clone in that case, and this is enough to invalidate the sufficient condition.

Let us consider now the sufficient condition in more detail. In the schema of the CNOT, in Diagram (27), we see that we have cloning for $b = 0$. Let us call cloning the function:

$$Clon: (a, 0) \rightarrow (a, a)$$

In the case with $a = 1$:

$$Clon \equiv (id, NOT): (1, 0) \rightarrow (1, 1)$$

In the case with $a = 0$:

$$Clon \equiv (id, id): (0, 0) \rightarrow (0, 0)$$

Let us consider the case with $a = 1$. See the following diagram

$$\begin{array}{ccc} L_m \text{ (1,0)} & & \\ \uparrow \sigma & \searrow \text{Clon} \equiv (id, NOT) & \\ L_0 \text{ (A, A}^\perp) & \xrightarrow{(F, F')} & L_m \times L_m \text{ (1,1)} \end{array} \quad (29)$$

With:

$$Clon: L_m \rightarrow L_m \times L_m, \quad Clon \equiv (id, NOT) \equiv \begin{cases} id: 1 \rightarrow 1 \\ NOT: 0 \rightarrow 1 \end{cases}$$

$$\sigma: L_0 \rightarrow L_m, \quad \sigma: \begin{cases} A \rightarrow 1 \\ A^\perp \rightarrow 0 \end{cases}$$

$$(F, F'): L_0 \rightarrow L_m \times L_m$$

$$\begin{aligned}
& F: A \rightarrow 1 \\
& F': A \perp \overset{\perp}{\rightarrow} A \xrightarrow{F} 1 \quad F' = F \circ \perp \\
& \left\{ \begin{array}{l} \sigma \circ id = F \\ \sigma \circ NOT = F \circ \perp \end{array} \right. \Rightarrow F = \sigma
\end{aligned} \tag{30}$$

The case with $a = 0$ is very similar, and is left as an exercise for the willing reader. Equation (30) says that if it exists a function $Clon$ in the metalanguage, such that the diagram (29) commutes, we recover the usual definition of self-reference, even if the corresponding logical connective in the object-language (in this case “times”) is non-idempotent, and we cannot skip self-referential paradoxes. Then, to be sure that a generalized version of self-reference is present in a given logic, one must check not only that the connective in the object-language is non-idempotent, but also that the corresponding physical link in the metalanguage (in this case the tensor product) does not lead to cloning in some particular case. In a sense, the “reflection” of the metalanguage into the object-language is in general incomplete, as far as self-reference is concerned. This is not the case, however, for the physical link of entanglement, because entangling a basis state with itself is physically meaningless (entanglement is a particular kind of superposition). On the other hand, as we have already seen, entangling an (unknown) quantum state with itself is forbidden by the no-cloning theorem. Then, in the case of entanglement, the reflection between the metalanguage and the object language is complete, and one can adopt the generalized version of self-reference.

Our conclusion is that Basic logic is the unique formal language that can reflect a metalanguage, like that of a quantum computer in a entangled state, which is never self-referential (and hence is completely paradox-free). Differently stated, the halting problem in quantum computing appears meaningless once the adequate logical language (Basic logic) is adopted.

Appendix A

In this appendix we give a formal proof of Metatheorem 1 discussed in Section 3. We try proving $Q_A @ Q_A \doteq Q_A$.

Let us try first $Q_A @ Q_A \vdash Q_A$.

There are no rules of Basic logic that we can use in the derivation, which can lead to a proof:

$$\overline{Q_A @ Q_A \vdash Q_A}$$

And, as the cut-elimination theorem holds in Basic logic [11], we are sure that there are no other rules leading to a proof.

Let us try now the other way around: $Q_A \vdash Q_A @ Q_A$.

The only rule we can use in the derivation is the @-formation rule, and there are no further rules in Basic logic, which would lead to a proof:

$$\frac{Q_A \vdash Q_A, Q_A}{Q_A \vdash Q_A @ Q_A} @ - form.$$

And, again, because of cut-elimination, we are sure that there are no other rules leading to a proof. For the sake of the physical interpretation, we show now that in the case the contraction and weakening rules did hold, the proof would be possible.

Let us prove first $Q_A @ Q_A \vdash Q_A$.

$$\frac{\frac{Q_A \vdash Q_A \quad Q_A \vdash Q_A}{Q_A @ Q_A \vdash Q_A, Q_A} @ - \text{expl.refl.}}{Q_A @ Q_A \vdash Q_A} \text{contr.}$$

Let us prove now the other way around $Q_A \vdash Q_A @ Q_A$.

$$\frac{\frac{Q_A \vdash Q_A}{Q_A \vdash Q_A, Q_A} \text{weak.}}{Q_A \vdash Q_A @ Q_A} @ - \text{form.}$$

It is impossible to prove $Q_A \vdash Q_A @ Q_A$ in Basic logic, because the weakening rule (in the step *weak.*) does not hold. In conclusion, it is impossible to prove idempotence of @ in Basic logic, because of the absence of the two structural rules of weakening and contraction.

References

- [1] P. A. Zizzi. Basic logic and quantum entanglement. arXiv: quant-ph/0611119, submitted for publication in the Proceedings of DICE 2006 (Journal of Physics: Conference Series, Publ. Institute of Physics, London).
- [2] G. Sambin, G. Battilotti, and C. Faggian. Basic logic: reflection, symmetry, visibility. *J. Symbolic Logic*, 65(3):979–1013, 2000.
- [3] G. Priest. Paraconsistent logic. In *Handbook of philosophical logic (Second Edition)*, Vol. 6, pages 287–393. Kluwer Acad. Publ., Dordrecht, 2002.
- [4] R. Josza and N. Linden. On the role of entanglement in quantum computational speed-up. arXiv: quant-ph/0201143.
- [5] W. K. Wootters and W. H. Zureck. A single quantum cannot be cloned. *Nature*, 299:802, 1982.
- [6] A. K. Pati and Braunstein S. L. Impossibility of deleting an unknown quantum state. *Nature*, 404:164, 2000. arXiv: quant-ph/9911090.
- [7] M. A. Nielsen and I. L. Chuang. *Quantum computation and quantum information*. Cambridge University Press, Cambridge, 2000.
- [8] J. S. Bell. *Speakable and unspeakable in quantum mechanics*. Cambridge University Press, Cambridge, 1987. Collected papers on quantum philosophy.
- [9] J. Y. Girard. Linear logic. *Theoret. Comput. Sci.*, 50(1):1–102, 1987.
- [10] A. Tarski. *Logic, semantics, metamathematics*. Hackett Publishing Co., Indianapolis, second edition, 1983. Papers from 1923 to 1938, Translated by J. H. Woodger, Edited and with an introduction by John Corcoran.
- [11] C. Faggian and G. Sambin. From basic logic to quantum logics with cut-elimination. In *Proceedings of the International Quantum Structures Association 1996 (Berlin)*, volume 37, pages 31–37, 1998.